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ON CERTAIN SINGULARITIES IN SOLUTIONS OF EQUATIONS OF BOUNDARY LAYER ON A MOVING SURFACE*

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Singularities of solutions of equations for unsteady boundary layer, for the boundary layer on a moving surface, and for a wake generated by a specified positive pressure gradient are investigated.

Investigations of unsteady boundary layer separation on a solid surface led to the establishment of the known analogy of that phenomenon with the steady separation on a moving surface /1,2/. According to the Moore-Rotta-Sears definition (see, e.g., /1,2/) the simultaneous vanishing of friction and of the velocity vector longitudinal component occurs at the point of boundary layer separation on a moving surface /1,2/. That point is singular and lies inside the boundary layer /1—4/. The appearance of the singularity is similar to that of the Goldstein singularity /5/ at the point of zero friction of the moving surface (see, also, the survey in /6/). Although separation is actually self-induced /7,8/, investigation of the structure of singularities which, generally and unavoidably, appear in solutions of equations of boundary layer with specified positive pressure gradient is in itself of interest. Appearance of such singularities, for instance in numerical solutions /3,4/ always indicates the impossibility of laminar flow.

1. Let us consider the behavior of solutions of equations of laminar boundary layer in a two-dimensional flow of incompressible fluid with a specified positive pressure gradient on a finite section of a surface moving in the downstream direction at constant velocity. We denote by Lx and Ly the curvilinear orthogonal coordinates directed, respectively, along the body surface and the normal to it, by $U_\infty u$ and $U_\infty v$ the corresponding velocity vector projections, by $p_\infty + \rho U_\infty^2 p$ the pressure, and by ρ the density. L is a characteristic dimension of the body in the stream, U_∞ and p_∞ are parameters of the oncoming stream, and $R = U_\infty L / \nu$, $R \to \infty$ is the Reynolds number.

We express the equations of the boundary layer and the respective boundary conditions in terms of Mises variables

$$u \frac{\partial u}{\partial x} + \frac{dp}{dx} = u \frac{\partial}{\partial \Psi}\left(u \frac{\partial u}{\partial \Psi}\right), \quad \frac{\partial Y}{\partial \Psi} = \frac{1}{u}, \quad u \frac{\partial Y}{\partial x} = V \tag{1.1}$$

$$u = U_w, \quad Y = V = 0, \quad (\Psi = 0), \quad u \to U(x) \ (\Psi \to \infty)$$

$$u = U^*(\Psi) \quad (x = x_0 < 0), \quad \Psi = R^{1/2}\psi, \quad Y = R^{1/2}y, \quad V = R^{1/2}v$$

where ψ is the dimensionless stream function, $U_w = O(1)$, $U_w > 0$ is the velocity of the body surface, and $U^*(\Psi)$ is the initial velocity profile. Function $U(x)$ which determines velocity as $\Psi \to \infty$ is assumed known from the solution of the external problem which in the neighborhood of point $x = 0$ can be represented as

$$U(x) = a_0 + a_0^{-1}\lambda_0(-x) + O[(-x)^2] \quad (x \to -0) \tag{1.2}$$

The pressure gradient behavior is then defined as follows:

$$dp / dx = \lambda_0 + O[(-x)] \quad (x \to -0) \tag{1.3}$$

where a_0 and λ_0 are positive constants.

Let us consider the behavior of solution of the boundary layer equations in the neighborhood of the point with coordinates $(0, Y_0)$, where Y_0 is some constant. Let this be an arbitrary point, i.e. both components of the velocity vector are at that point finite. Then the solution of problem (1.1)—(1.3) as $x \to -0$ and $Y \to Y_0$, can be represented in terms of usual variables in the form of the regular expansion

$$\Psi = \Psi_0(Y) + (-x)\Psi_1(Y) + O[(-x)^2], \quad \Psi_0(Y) = \Psi_s + \alpha_1 Y^* + \alpha_2 Y^{*2} + \alpha_3 Y^{*3} + \alpha_4 Y^{*4} + O(Y^{*5}) \tag{1.4}$$

$$\Psi_1(Y) = \beta_0 + \beta_1 Y^* + \beta_2 Y^{*2} + \beta_3 Y^{*3} + O(Y^{*4}), \quad Y^* = Y - Y_0$$

Then $\Psi_s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0$ are arbitrary constants and constants $\beta_1, \beta_2, \beta_3$ are defined in terms of the latter and of λ_0 from (1.3).

If at the considered point the terms that define friction and the velocity vector longitudinal component simultaneously vanish, then in (1.4)

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$$\alpha_1 = \alpha_2 = 0, \quad \alpha_3 = \lambda_0 / 6, \quad \beta_0 = 24\alpha_4\lambda_0^{-1} \tag{1.5}$$

Let us now consider problem (1.1)—(1.3) for the neighborhood of the point at which friction and the longitudinal velocity vector are zero, assuming that relations (1.5) between constants are not satisfied and, consequently, the solution is irregular. This is a more general case since in the solution outside the neighborhood of that point constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0$ are arbitrary and determined by the external boundary conditions, while in the neighborhood of the point the solution is regular, when conditions (1.5) are imposed on these constants.

If the boundary layer is in the region of increasing pressure downstream, the fluid in that layer is retarded. Since the body surface moves downstream, hence, beginning at some $x = x^* > x_0$ (Fig.1) the velocity profile along each line $x = \text{const}$ has a minimum at some point lying outside the body surface, and that point is the point of zero friction. Hence in the neighborhood of point $x = -0$ friction vanishes along some line.

As in /5/, we introduce in the neighborhood of the zero friction line a sublayer in which inertial and viscous terms are quantities of the same order. In the orthogonal system of coordinates (x_1, Y_1) attached to the zero friction line and with origin at point $(0, \Psi_s)$ the solution for such sublayer (region *1* in Fig.2)

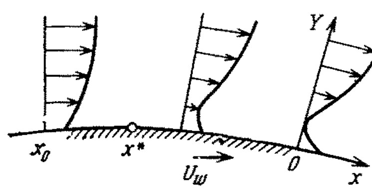

Fig.1

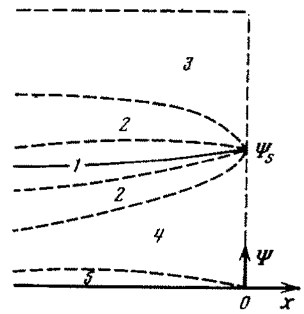

Fig.2

is of the form

$$\Psi = \Psi_s + (-x_1)^{3/4} f_0(\eta) + (-x_1)^{\alpha} f_1(\eta) + O[(-x_1)^{7/4}], \quad \eta = Y_1(-x_1)^{-1/4}, \quad \alpha \in (3/4, 7/4) \tag{1.6}$$

Substituting expansions (1.6) into (1.1)—(1.3), we obtain the following problem:

$$f_0''' - \frac{3}{4} f_0'' f_0 + \frac{1}{2} f_0'^2 - \lambda_0 = 0; \quad f_0(0) = c_0, \quad f_0''(0) = 0 \tag{1.7}$$

with $f_0(\eta)$ free of exponentially increasing terms as $|\eta| \to \infty$.

At least two solutions of this problem exist. One of these, obtained in /5/, is of the form

$$f_0(\eta) = \lambda_0\eta^3 / 6 \quad (c_0 = 0) \tag{1.8}$$

and the other of the form

$$f_0(\eta) = (2\lambda_0)^{1/2}\eta + c_0 \tag{1.9}$$

Let us consider in detail the solution of the problem corresponding to solution (1.9). (The case when the solution of problem (1.7) is of the form (1.8) and, consequently, satisfies the supplementary condition $f_0'(0) = 0$, is considered in Sect.3, below). For the function $f_1(\eta)$ of expansion (1.6) with (1.9) taken into account, we obtain the following problem:

$$f_1'(\eta) = \Phi(\eta_1), \quad \eta = 2\cdot 3^{-1/2}(2\lambda_0)^{-1/4}\eta_1 - c_0(2\lambda_0)^{-1/2}$$
$$\nu_0 = (4\alpha + 1)/3, \quad \Phi'' - \eta_1\Phi' + \nu_0\Phi = 0, \quad \Phi'(0) = 0 \tag{1.10}$$

where $\Phi(\eta_1)$ does not contain exponentially increasing terms as $|\eta_1| \to \infty$. The obtained equation for function $\Phi(\eta_1)$ is the Weber equation (see /9/). Problem (1.10) has a zero solution only when $\nu_0 = 2(n-1)$, $n = 1, 2, \ldots$, i.e. $\alpha = (6n - 7)/4$. Since in expansion (1.6) $\alpha \in (3/4, 7/4)$, we have $\alpha = 5/4$ and the solution is $\Phi(\eta_1) = c_1(\eta_1^2 - 1)$. Reverting to the initial variable η and satisfying the condition that friction must vanish when $\eta = 0$, we obtain

$$f_1(\eta) = c_1\eta[(2\lambda_0)^{1/2}\eta^2/4 - 1] + c_2, \quad c_0 = 0 \tag{1.11}$$

where c_1 and c_2 are arbitrary constants.

Note that solution (1.6), (1.9), (1.11) obtained for region *1* cannot be merged with the solution for the basic part of the boundary layer, where the velocity vector longitudinal component is a quantity of order unity. Because of this, we introduce one more sublayer (region *2* in Fig.2), where the flow is locally inviscid. The respective equations are of the form

$$u = (-x)^{1/2} g_0(\xi) + O[(-x)^{2(1-\beta)}] + O[(-x)^{3/2}], \quad \xi = \chi(-x)^{-\beta}, \quad \chi = \Psi - \Psi_s + \varphi_0(x), \quad \beta < {}^3/_4 \qquad (1.12)$$

where $\varphi_0(x) \to 0$ as $x \to -0$, and which defined the difference between the zero friction line and the streamline $\Psi = \Psi_s$. Substituting expansion (1.12) into (1.1)—(1.3) and setting $\varphi_0(x) = k_0(-x)^\beta + o[(-x)^\beta]$, we obtain for function $g_0(\xi)$ the equation

$$2\beta(\xi - k_0) g_0' g_0 - g_0^2 + 2\lambda_0 = 0$$

Integration of this equation and its merging with the solution for region *1* yields $\beta = {}^1/_2$, $k_0 = 0$, $g_0(\xi) = (A_0^2\xi^2 + 2\lambda_0)^{1/2}$. Having determined constant β, we obtain for the first two terms of expansion in region *2*, as $x \to -0$ the final solution of the form

$$u = (-x)^{1/2} g_0(\xi) + (-x) g_1(\xi) + O[(-x)^{3/2}], \quad Y = G^*(x) + h_0(\xi) + (-x)^{1/2} h_1(\xi) + O[(-x)] \qquad (1.13)$$

$$\xi = \chi(-x)^{-1/2}, \quad \chi = \Psi - \Psi_s + \varphi_0(x), \quad \varphi_0(x) = o[(-x)]$$

$$g_0(\xi) = (A_0^2\xi^2 + 2\lambda_0)^{1/2}, \quad g_1(\xi) = A_1\xi^3(A_0^2\xi^2 + 2\lambda_0)^{-1/2} - A_0^2(3\lambda_0)^{-1}(A_0^2\xi^2 + 2\lambda_0)$$

$$h_0(\xi) = (2A_0)^{-1}\ln\left[\frac{g_0(\xi) + A_0\xi}{g_0(\xi) - A_0\xi}\right], \quad h_1(\xi) = \frac{A_0^2}{3\lambda_0}\xi - \frac{A_1(A_0^2\xi^2 + 4\lambda_0)}{A_0^4(A_0^2\xi^2 + 2\lambda_0)^{1/2}}$$

where $G^*(x)$ is some, so far arbitrary, function. The arbitrary constants A_0, A_1, Ψ_s are determined by the external boundary conditions. (Formulas for the velocity vector vertical component are directly obtained from the third of Eqs. (1.1), and are not given here).

Solution (1.13) for region *2* satisfies the condition that friction must vanish along line $\chi = 0$ and be regular in its neighborhood. It is, therefore, superfluous to introduce region *1*, since the solution in it is a continuation of solution (1.13) in the region where Ψ is $\Psi_s = O[(-x)^{3/4}]$.

In regions *3* and *4* (Fig.2), where the velocity vector longitudinal component is a quantity of order unity; in conformity with (1.2), as $x \to -0$, the solution can be represented in the form

$$u = U_i(\Psi) + (-x) U_i^*(\Psi) + O[(-x)^2], \quad Y = G_i(x) + Y_i(\Psi) + (-x) Y_i^*(\Psi) + O[(-x)^2] \qquad (1.14)$$

$$U_i^*(\Psi) = \lambda_0 U_i^{-1}(\Psi) - [U_i(\Psi) U_i'(\Psi)]', \quad Y_i(\Psi) = \int U_i^{-1}(\Psi)\, d\Psi, \quad Y_i^*(\Psi) = -\int U_i^*(\Psi) U_i^{-2}(\Psi)\, d\Psi$$

$$i = 3, 4, \quad U_3(\Psi) \to a_0 \quad (\Psi \to \infty)$$

whose merging with the solution in region *2* yields

$$\begin{aligned} &U_3(\Psi) = A_0(\Psi - \Psi_s) + O[(\Psi - \Psi_s)^2] \quad (\Psi \to +\Psi_s), \quad A_0 > 0 \\ &G_3(x) = G^*(x) - (2A_0)^{-1}\ln[\lambda_0(2A_0^2)^{-1}(-x)] + o(1) \\ &U_4(\Psi) = A_0(\Psi_s - \Psi) + O[(\Psi_s - \Psi)^2] \quad (\Psi \to -\Psi_s) \\ &G_4(x) = G^*(x) + (2A_0)^{-1/2}\ln[\lambda_0(2A_0^2)^{-1}(-x)] + o(1) \end{aligned} \qquad (1.15)$$

To satisfy boundary conditions in (1.1) when $\Psi = 0$ we introduce the sublayer (region *5* in Fig.2) in which the effect of viscosity forces will be assumed substantial. We represent the solution in the form

$$u = U_w + (-x)^\gamma F_0(\zeta) + O[(-x)], \quad \zeta = \Psi(-x)^{-1/2}, \ \gamma \in (0, 1)$$

The substitution of this expansion into (1.1)—(1.3) shows that $F_0(\zeta)$ satisfies the Weber equation $2U_w F_0'' - \zeta F_0' + 2\gamma F_0 = 0$ with the boundary conditions that $F_0(0) = 0$ and that $F_0(\zeta)$ does not contain exponentially increasing terms as $\zeta \to \infty$. The problem has a nonzero solution only if $\gamma = n - {}^1/_2$, $n = 1, 2, \ldots$. Since $\gamma \in (0, 1)$, $\gamma = {}^1/_2$ and then $F_0(\zeta) = b_0\zeta$. Finally, the solution in region *5*, which satisfies the boundary conditions for $\Psi = 0$, can be represented in the form

$$u = U_w + (-x)^{1/2} F_0(\zeta) + (-x) F_1(\zeta) + O[(-x)^{3/2}], \quad Y = (-x)^{1/2} U_w^{-1}\zeta + (-x) H_0(\zeta) + O[(-x)^{3/2}] \qquad (1.16)$$

$$\zeta = \Psi(-x)^{-1/2}, \ F_0(\zeta) = b_0\zeta, \ F_1(\zeta) = b_1\zeta^2, \ H_0(\zeta) = -b_0(2U_w^2)^{-1}\zeta^2, \ b_1 = (\lambda_0 - b_0^2 U_w)(2U_w^2)^{-1}$$

The arbitrary constant b_0 is determined by external boundary conditions. The merging of solutions (1.16) with solution (1.14), (1.15) shows that for region *4* we have $U_4(\Psi) = U_w + b_0\Psi + b_1\Psi^2 + O(\Psi^3)$ as $\Psi \to 0$ and $G_4(x) = 0$. The last equality together with formula (1.15) yields for function $G^*(x)$ which appears in (1.13)—(1.15) the expansion

$$G^*(x) = -(2A_0)^{-1}\ln[\lambda_0(2A_0^2)^{-1}(-x)] + o(1) \quad (x \to -0) \qquad (1.17)$$

Thus the boundary layer thickness logarithmically increases as point $x = -0$ is approached. Owing to this, the solution of boundary layer equations in the neighborhood of that point behave in a singular manner. It follows from (1.16) that friction on the body surface is finite quantity

$$\tau_w = u \frac{\partial u}{\partial \Psi}\Big|_{\Psi=0} = b_0 U_w + O[(-x)]$$

2. Let us show that the derived solution cannot be continuously extended beyond point $x = 0$.

In conformity with /5/ we continue the solution for the basic part of the boundary layer into the region of positive x. As $x \to +0$, the solution there is of the form

$$u = U_3(\Psi) + xU_3^\circ(\Psi) + O(x^2), \quad U_3^\circ(\Psi) = -\lambda_0 U_3^{-1}(\Psi) + [U_3(\Psi)\, U_3'(\Psi)]' \qquad (2.1)$$

For $\Psi \to +\Psi_s$ $U_3(\Psi) = A_0(\Psi - \Psi_s) + O[(\Psi - \Psi_s)^2]$. Hence expansion (2.1) is not valid in the region where $\Psi - \Psi_s = O(x^{1/2})$. The solution in that region can be represented in the form

$$u = x^{1/2} R_0(s) + O(x), \; s = \chi^\circ x^{-1/2}, \; \chi^\circ = \Psi - \Psi_s + o(x)$$

Substituting this expansion into the boundary layer equation, taking into account (1.3), and integrating, we obtain

$$R_0(s) = (A_0^2 s^2 - 2\lambda_0)^{1/2}$$

which shows that when $s^2 < 2\lambda_0 A_0^{-2}$ the solution is imaginary. Thus the derived solution, similarly to Goldstein's solution for a stationary surface /5/ cannot be continuously extended beyond the singular point.

We recall that passing in the derived here equation from the coordinate system attached to the singular point to a system attached to the moving surface yields the solution for unsteady equations of the boundary layer (see /1,2/), which corresponds to the neighborhood of a singular point moving upstream at constant velocity $U_s = -U_w$.

3. Let us consider the case when the solution of problem (1.7) for function $f_0(\eta)$ is defined by formula (1.8) and, consequently, satisfies the supplementary condition $f_0'(0) = 0$.

The solution in the neighborhood of the considered point, obtained in /10/ is in this case of the form

$$\Psi = \Psi_s + (-x_1)^{3/4} f_0(\eta) + (-x_1) f_1^*(\eta) + (-x_1)^{5/4} f_2^*(\eta) + O[(-x_1)^{3/2}] \qquad (3.1)$$

$$\eta = Y_1(-x_1)^{-1/4}, \quad f_0(\eta) = \lambda_0 \eta^3 / 6, \quad f_1^*(\eta) = \alpha_4(\eta^4 + 24\lambda_0^{-1}) + a_1^* \eta^2$$

$$f_2^*(\eta) = \alpha_5 \eta^5 + 8 a_1^* \alpha_4 \lambda_0^{-1} \eta^3 + a_2^* \eta^2 + [120\alpha_5 \lambda_0 + 2 a_1^{*2} \lambda_0 - (24\alpha_4)^2] \lambda_0^{-2} \eta$$

where $\alpha_4, \alpha_5, a_1^*, a_2^*$ are arbitrary constants. If the line $Y_1 = 0$ does not coincide with that of zero friction, solution (3.1) has a singularity at point $x_1 = -0$. However with the use of Prandtl's transform which in this case is of the form

$$\Psi \to \Psi, \; x_1 \to x_1, \; Y_1 \to Y_1 - 2a_1^* \lambda_0^{-1}(-x_1)^{1/2} + O[(-x_1)^{3/4}]$$

the singularity can be eliminated. This transformation is similar to passing to a system of coordinates attached to the zero friction line. Hence it is necessary to set $a_1^* = a_2^* = 0$ in solution (3.1) which represents a regular expansion of (1.4) and (1.5) that can be continuously extended in to the region of positive x.

Thus solution (1.8) corresponds to a regular expansion in the neighborhood of the considered point. Solution (1.9) then corresponds to the singular behavior. The surmise in /1/ that solution (3.1) obtained in /10/ defines the singular behavior of the solution of problem (1.1) — (1.3) does not on the evidence of the above analysis appear to correspond to reality.

Note that when the solution is singular, friction in the first two terms of the expansion vanishes (as in /11/) along the streamline $\Psi = \Psi_s$. If, however, the solution is regular, friction vanishes along a line which generally is not a streamline. This is related to that the singular solution is locally inviscid and, consequently, vorticity remains along streamlines $\Psi_{YY} = \omega(\Psi)$.

4. Finally, let us consider the singularities originating on the axis of symmetry of a laminar wake in an incompressible fluid with a specified positive pressure gradient.

The flow in a two-dimensional steady symmetric wake downstream of the body the flow over which is laminar, is defined by equations of the boundary layer. As in Sect.1, we introduce dimensionless functions and independent variables, and locate the origin of a Cartesian coordinate system at the point where the axial velocity is zero. The boundary conditions at the wake axis of symmetry are of the form

$$u\frac{\partial u}{\partial \Psi} = Y = V = 0 \quad (\Psi = 0)$$

The solution for the flow outside the wake is assumed known, and velocity and pressure variations at the outer boundary of the wake as $x \to -0$ are defined by formulas (1.2) and (1.3).

On the basis of the above analysis of solution of the equation of the boundary layer on a moving surface it is possible to show that in the considered point neighborhood the solution generally behaves in a singular manner. In the inner region the solution is defined by formula (1.13) and in the wake basic part by formulas (1.14) and (1.15) in which we set $\Psi_s = \varphi_0(x) = A_1 = G^*(x) = 0$. This solution cannot be continuously extended beyond the singular point. If that point is regular, the solution in its neighborhood is defined by formulas (3.1) in which we set $\Psi_s = \alpha_4 = a_1^* = a_2^* = 0$. It can then be continuously extended in the region of positive x.

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